Our intent is to develop a 2D axisymmetric code for gravitational collapse. Several people have, of course, done this in the past. One of the most promising approaches was that pioneered by Nakamura and his collaborators. The theoretical framework they used was one based on work by Geroch and which they called the (2+1)+1 decomposition. The basic idea is to divide out the symmetry and perform a foliation of the resulting 3 dimensional manifold *a la* ADM. In the hopes that this will be a useful introduction to this formulation and the resulting equations, we will try to develop all of the machinery *ab initio*.

To begin, let us assume that we have an $n+1$ dimensional manifold. (We will be interested of course in the case that $n+1=4$, but we can do things somewhat generally at this point.) We also assume the existence of a Killing vector

$$\xi = \frac{\partial}{\partial\varphi},$$

where we have let the coordinate $x^{n+1}=\varphi$. For definiteness and in anticipation of our (2+1)+1 reduction in the presence of axisymmetry, one can think of this Killing vector as being spacelike and possessing closed orbits. For the moment, however, we will consider a slightly more general case, namely that the Killing vector is only non-null.

We want to divide out the action of the Killing vector. In mathematical parlance (and in the particular case that the Killing vector is spacelike with closed orbits), we are interested in the quotient space

$$M/S^1$$

where S^1 represents the topology of a circle. From a practical point of view we construct this space by projecting onto an n dimensional manifold. To this end, we define the projection operator (which will become the metric on the quotient manifold)

$$g_{\mu\nu} = \gamma_{\mu\nu} - \frac{1}{\kappa s^2} X_\mu X_\nu$$

where $\gamma_{\mu\nu}$ is the metric on the $n+1$ manifold, $X^\mu = \delta^\mu_\varphi$, is the Killing vector field and

$$X^\mu X_\mu = \kappa s^2$$

is the norm of the Killing vector and $\kappa = \pm 1$ depending on whether it is spacelike or timelike respectively.*

The inverse of this operator is

$$g^{\mu\nu} = \gamma^{\mu\nu} - \frac{1}{\kappa s^2} X^\mu X^\nu.$$

We want to construct the relevant quantities on the $n+1$ manifold in terms of n dimensional quantities. First, we make the following definitions:

$$A_\mu = \frac{1}{\kappa s^2} X_\mu, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

We can write the metric on the $n+1$ manifold as

$$\gamma_{\mu\nu} = \begin{pmatrix} g_{ab} + \kappa s^2 A_a A_b & s^2 A_a \\ s^2 A_b & s^2 \end{pmatrix}$$

Using these relations, the connection beomes

$$\begin{aligned} {}^{(n+1)}\Gamma^\lambda_{\mu\nu} &= {}^{(n)}\Gamma^\lambda_{\mu\nu} + \kappa \Omega^\lambda_{\mu\nu} \\ &= {}^{(n)}\Gamma^\lambda_{\mu\nu} + \frac{\kappa}{2} s^2 g^{\lambda\sigma}\left[A_\nu F_{\mu\sigma} + A_\mu F_{\nu\sigma} - \partial_\sigma(\ln s^2) A_\mu A_\nu\right] \\ &\quad + \frac{\kappa}{2} A^\lambda \left[\partial_\mu(s^2 A_\nu) + \partial_\nu(s^2 A_\mu)\right]. \end{aligned}$$

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* We will use Greek indices such as $\mu, \nu, \lambda, \dots$ to run from 1 to $n+1$ and lowercase Latin indices such as $a, b, c, \dots$ to run from 1 to n. When we apply the ADM formalism to our n dimensional manifold, we will use uppercase Latin indices such as $A, B, C, \dots$ to run from 1 to $n-1$.

It is worth mentioning that the metric $\gamma_{\mu\nu}$ lowers indices and $\gamma^{\mu\nu}$ raises indices. As a result, a useful identity in deriving the following is

$$g^{\mu\nu}A_\mu = 0.$$

A few other handy identities are*

$$\begin{aligned} A^\nu A_{\nu,\mu} &= 0 \\ A^\nu \partial_\nu(\log s^2) &= 0 \\ A^\nu F_{\nu\mu} &= 0. \end{aligned}$$

The $n+1$ dimensional volume element can be written

$$\sqrt{-\gamma} = s\sqrt{-g}.$$

This can be got from the relation

$$^{(n+1)}\Gamma^{\nu}_{\mu\nu} = \partial_\mu(\ln\sqrt{-\gamma}).$$

The Ricci tensor can be written

$$^{(n+1)}R_{\mu\nu} = {}^{(n)}R_{\mu\nu} + \kappa\, {}^{(n)}\nabla_\lambda \Omega^\lambda_{\mu\nu} - \kappa\, {}^{(n)}\nabla_\mu \Omega^\lambda_{\nu\lambda} + \Omega^\lambda_{\mu\nu}\partial_\lambda(\ln s) - \Omega^\sigma_{\mu\lambda}\Omega^\lambda_{\nu\sigma}$$

where ${}^{(n)}\nabla_\mu$ is the covariant derivative on the n dimensional manifold and we have used $\Omega^\sigma_{\sigma\lambda} = \partial_\lambda(\ln s)$. We now want the components of ${}^{(n+1)}R_{\mu\nu}$ in terms of the n-dimensional quantities. Expressed in terms of the n dimensional fields, these become

$$\begin{aligned} {}^{(n+1)}R_{\varphi\varphi} &= \frac{1}{4}s^4 F_{bc}F^{bc} - \kappa s\, {}^{(n)}\nabla^a\, {}^{(n)}\nabla_a s \\ {}^{(n+1)}R_{\varphi a} &= \frac{\kappa}{2s}\, {}^{(n)}\nabla^c(s^3 F_{ac}) + A_a\left[\frac{1}{4}s^4 F_{bc}F^{bc} - \kappa s\, {}^{(n)}\nabla^a\, {}^{(n)}\nabla_a s\right] \\ {}^{(n+1)}R_{ab} &= {}^{(n)}R_{ab} - \frac{1}{s}\, {}^{(n)}\nabla_a\, {}^{(n)}\nabla_b s - \frac{\kappa}{2}s^2 F_{ac}F_b{}^c \\ &\quad - \frac{\kappa}{s}\, {}^{(n)}\nabla^c\left[s^3 F_{c(a}\right]A_{b)} + A_a A_b\left[\frac{1}{4}s^4 F_{bc}F^{bc} - \kappa s\, {}^{(n)}\nabla^a\, {}^{(n)}\nabla_a s\right]. \end{aligned}$$

The Ricci scalar is found to be

$$^{(n+1)}R = {}^{(n)}R - \frac{2}{s}\, {}^{(n)}\nabla^a\, {}^{(n)}\nabla_a s - \frac{1}{4}s^2 F_{bc}F^{bc}.$$

which in the absence of matter is consistent with contracting on ${}^{(n)}R_{ab}$ above and using the other equations.

Everything up to now has been relatively general. Let's now choose $n+1=4$ and assume axisymmetry ($\kappa = 1$). The theory we want to consider is Einstein gravity coupled to some fundamental matter field. An example of the matter would be the harmonic map model. The existence of the axisymmetry motivates our use of the Kaluza-Klein like reduction. The action for this theory is then given by

$$S = \int d^4x\sqrt{-g}\left\{\frac{R}{16\pi G} - \frac{|\nabla f|^2}{(1-\kappa|f|^2)^2}\right\},$$

where f is a complex scalar field and κ is now *not* the same as before (that $\kappa = 1$ since we are assuming a spacelike Killing vector), but parameterizes a family of theories. The equations of motion for this are

$$\begin{aligned} G_{\mu\nu} &= 8\pi G T_{\mu\nu} \\ &= \frac{8\pi G}{(1-\kappa|f|^2)^2}\left(\nabla_\mu f\nabla_\nu f^* + \nabla_\mu f^*\nabla_\nu f - g_{\mu\nu}|\partial f|^2\right), \\ \nabla^\mu\nabla_\mu f &= \frac{-2\kappa f^*}{1-\kappa|f|^2}\nabla_\mu f\nabla^\mu f. \end{aligned}$$

* Note however that $A_\nu A^\nu{}_{,\mu} \neq 0$.

We can also write the Einstein equations a bit more succinctly as

$$R_{\mu\nu} = \frac{8\pi G}{(1-\kappa|f|^2)^2}\left(\nabla_\mu f \nabla_\nu f^* + \nabla_\mu f^* \nabla_\nu f\right)$$

where we have changed notation slightly and are now using ∇_μ as the covariant derivative on the 4 dimensional manifold with the metric $\gamma_{\mu\nu}$. Assuming axisymmetry for the system and that our prior reduction holds, we have

$$\begin{aligned} D^a D_a s &= \frac{1}{4} s^3 F_{bc} F^{bc} \\ D^c(s^3 F_{ac}) &= 0 \\ R_{ab} &= \frac{1}{s} D_a D_b s + \frac{1}{2} s^2 F_{ac} F_b{}^c + \frac{8\pi G}{(1-\kappa|f|^2)^2}\left(D_a f D_b f^* + D_a f^* D_b f\right) \\ D_a D^a f &= -\frac{1}{s} D_a s D^a f - \frac{2\kappa f^*}{1-\kappa|f|^2} D_a f D^a f. \end{aligned}$$

where D_a is the covariant derivative on the 3 dimensional manifold which possesses the matric g_{ab}. Note that in the reduction from 4 to 3 dimensions the D'Alembertian of the field f in 4 dimensions picks up a term involving s. Using our earlier relations, this is straightforward to show. The covariant derivative (gradient) of f in 4 dimensions reduces essentially unchanged to 3 dimensions since it is just partial differentiation and we assume there is no dependance of f on φ.

We can simplify these equations a bit further for our choice of the matter fields. The second equation defines locally a potential field w according to*

$$s^3 F_{ab} = -\epsilon_{abc} D^c w.$$

We can now make the following simplifications

$$\begin{aligned} F_{ac} F_b{}^c &= \frac{1}{s^6}\left(D_a w D_b w - g_{ab}(Dw)^2\right) \\ F_{ab} F^{ab} &= -\frac{2}{s^6}(Dw)^2 \end{aligned}$$

Also, the above divergence equation is now replaced with

$$D_a D^a w = \frac{3}{s} D_b s D^b w.$$

This follows from the identity $D_{[a} F_{bc]} = 0$. Note that we could do this only because of the nature of our choice of matter fields. For instance, Nakamura *et al* are unable to make this simplification because their use of perfect fluids as their matter yields a "source" term on the right hand side of their divergence equation.

Rewriting our equations in terms of this potential field w, we have

$$\begin{aligned} D_a D^a s &= -\frac{1}{2s^3}(Dw)^2 \\ D_a D^a w &= \frac{3}{s} D_b s D^b w \\ R_{ab} &= \frac{1}{s} D_a D_b s + \frac{1}{2s^4}\left(D_a w D_b w - g_{ab}(Dw)^2\right) + \frac{8\pi G}{(1-\kappa|f|^2)^2}\left(D_a f D_b f^* + D_a f^* D_b f\right) \\ D_a D^a f &= -\frac{1}{s} D_a s D^a f - \frac{2\kappa f^*}{1-\kappa|f|^2} D_a f D^a f. \end{aligned}$$

* w is also called the twist potential or the scalar twist of a Killing field. It is defined in Wald 7.4 as the function such that

$$\nabla_a w = w_a = \epsilon_{abcd} \xi^b \nabla^c \xi^d$$

where ξ^a is a Killing vector field. One can verify that indeed this is true in our case. [up to a constant factor of 2 I think]

Note a couple of things about these equations. One is that the field w never appears, only its derivatives. Also, in the case $w = 0$ (no rotation), we can define a field ϕ such that $s = e^{\phi}$, and in this case, ϕ does not appear anywhere, only it's derivatives. I mention this because I was under the impression that numerically this property is sometimes a "good thing." If not, I await being corrected.

Now we want to do the standard ADM reduction of the 3 dimensional space and consider the extra fields we have as essentially matter fields. Taking our nonlinear sigma model as an example we would then have in the 3 (or 2+1) dimensional space a complex scalar field f, and two real scalar fields s and w. These are coupled to each other in non-trivial ways, but nevertheless, we can think of them as simply matter.

The 2+1 decomposition proceeds as one would expect. We decompose g_{ab} as

$$d\sigma^2 = g_{ab}dx^a dx^b = -\alpha^2 dt^2 + h_{AB}(dx^A + \beta^A dt)(dx^B + \beta^B dt)$$

where α is the lapse and β^A is the shift vector. The indices A and B now range over 1 and 2. The metric g_{ab} can be written with the parameterization

$$g_{ab} = \begin{pmatrix} -\alpha^2 + h_{AB}\beta^A\beta^B & h_{AB}\beta^B \\ h_{AB}\beta^A & h_{AB} \end{pmatrix}$$

where we will use the parameterization

$$h_{AB} = \begin{pmatrix} a^2 + a^2b^2c^2 & a^2b^2c \\ a^2b^2c & a^2b^2 \end{pmatrix}$$

There are various gauge choices we could make, but we will try to keep things somewhat general for the moment. We make the usual definitions. We define a timelike vector n^a and the extrinsic curvature $K_{ab} = K_{ba} = -D_a n_b$. The usual matter quantities are defined

$$\begin{aligned} \rho &= n_a n_b T^{ab} \\ j_A &= -n_a T^a{}_A \\ S_{AB} &= T_{AB} \end{aligned}$$

with the 2-metric h_{AB} and its inverse lowering and raising indices in the 2 space.

The evolution equations for the 2 dimensional metric h_{AB} and the 2 dimensional extrinsic curvature K_{AB} can be written in a form analagous to the usual form, namely

$$\begin{aligned} \partial_t h_{AB} &= -2\alpha K_{AB} + \Delta_A\beta_B + \Delta_B\beta_A \\ \partial_t K_{AB} &= (\beta^C\Delta_C K_{AB} + K_{AC}\Delta_B\beta^C + K_{BC}\Delta_A\beta^C) \\ &\quad + \alpha[KK_{AB} + {}^{(2)}R_{AB}] - 2\alpha K_{AC}K_B{}^C - \Delta_A\Delta_B\alpha - \alpha\,{}^3R_{AB} \end{aligned}$$

where we have now defined $n_a = (-\alpha, 0, 0)$ (and hence $n^a = \frac{1}{\alpha}(1, -\beta^1, -\beta^2)$) and the covariant derivative on the 2 dimensional surfaces to be Δ_A. This derivative, of course, is with respect to the metric h_{AB}. Our task now is to express the remaining 3 dimensional quantities, such as ${}^3R_{AB}$ above and $D_a D^a s$, in terms of fields on the 2 dimensional slicings. To this end it is useful to quote some useful relations between the two types of dimensional quantities. One can show

$$\begin{aligned} h_A{}^a h_B{}^b D_a D_b s &= \Delta_A\Delta_B s + (n^a\partial_a s)K_{AB} \\ D_a s D^a w &= \Delta_A s \Delta^A w - (n^a\partial_a s)(n^b\partial_b w) \end{aligned}$$

where (in something of an abuse of notation) we mean what we say in all the above. Namely, the indices A and B range over 1 and 2 while a and b range over 0, 1, and 2. The point is that those terms which include the sum over a and b indices include time components.

In addition to the reduction of our s and w fields, we also need to know how the "real" matter goes from 3 to 2 dimensions. To this end, consider only the matter part of ${}^3R_{ab}$. In the slicings, it is given by

$$ {}^3R^{mat}_{AB} = 8\pi G\left[S_{AB} - \frac{1}{2}h_{AB}T\right] $$

where T is the trace of the (4-dimensional!) stress tensor and can be written as

$$ \begin{aligned} T &= \gamma_{\alpha\beta}T^{\alpha\beta} \\ &= [h_{\alpha\beta} - n_\alpha n_\beta + \frac{1}{s^2}X_\alpha X_\beta]T^{\alpha\beta} \\ &= S - \rho + \frac{1}{s^2}T_{\varphi\varphi} \end{aligned} $$

Note that for the particular choice of the nonlinear sigma model as our matter, the final line above is equal to $2T_{\varphi\varphi}/s^2$.

Okay, so let's write out these equations. The wave equations for the fields s and w can each be decomposed into two first order equations given as follows

$$ \begin{aligned} \partial_t s - \beta^A\partial_A s &= -\alpha s\chi \\ \partial_t\chi - \beta^A\partial_A\chi &= -\frac{\alpha}{s}\Delta_A\Delta^A s + \alpha\chi(K+\chi) \\ &\quad + \frac{\alpha}{2s^4}(u^2 - \Delta_A w\Delta^A w) - \frac{1}{s}h^{AB}\partial_A s\partial_B\alpha \\ \partial_t w - \beta^A\partial_A w &= -\alpha u \\ \partial_t u - \beta^A\partial_A u &= -\alpha\Delta_A\Delta^A w + \alpha u(K - 3\chi) \\ &\quad + \frac{3\alpha}{s}\Delta_A s\Delta^A w - h^{AB}\partial_A w\partial_B\alpha \end{aligned} $$

where we have made the definitions

$$ \begin{aligned} \chi &= -\frac{1}{s}n^a\partial_a s \\ u &= -n^a\partial_a w \end{aligned} $$

The time evolution of the extrinsic curvature is now

$$ \begin{aligned} \partial_t K_{AB} &= (\beta^C\Delta_C K_{AB} + K_{AC}\Delta_B\beta^C + K_{BC}\Delta_A\beta^C) \\ &\quad + \alpha[KK_{AB} + {}^{(2)}R_{AB}] - 2\alpha K_{AC}K_B{}^C - \Delta_A\Delta_B\alpha \\ &\quad - \frac{\alpha}{s}\Delta_A\Delta_B s + \alpha K_{AB}\chi - \frac{\alpha}{2s^4}\left[\Delta_A w\Delta_B w - h_{AB}(\Delta_C w\Delta^C w - u^2)\right] \\ &\quad - 8\pi G\alpha\left[S_{AB} - \frac{1}{2}h_{AB}(S - \rho + \frac{1}{s^2}T_{\varphi\varphi})\right] \end{aligned} $$

Raising one index on the extrinsic curvature and simplifying the shift vector terms, we can write this together with the evolution of the 2-metric as

$$ \begin{aligned} \partial_t h_{AB} &= -2\alpha K_{AB} + h_{BC}\partial_A\beta^C + h_{AC}\partial_B\beta^C + \beta^C\partial_C h_{AB} \\ \partial_t K_A{}^B &= K_C{}^B\partial_A\beta^C - K_A{}^C\partial_C\beta^B + \beta^C\partial_C K_A{}^B \\ &\quad + \alpha[KK_A{}^B + {}^{(2)}R_A{}^B] - \Delta_A\Delta^B\alpha \\ &\quad - \frac{\alpha}{s}\Delta_A\Delta^B s + \alpha K_A{}^B\chi - \frac{\alpha}{2s^4}\left[\Delta_A w\Delta^B w - h_A{}^B(\Delta_C w\Delta^C w - u^2)\right] \\ &\quad - 8\pi G\alpha\left[S_A{}^B - \frac{1}{2}h_A{}^B(S - \rho + \frac{1}{s^2}T_{\varphi\varphi})\right] \end{aligned} $$

Let us now consider the constraint equations. They stem from the Gauss-Codazzi relations. For the embedding of a two dimensional surface in a three dimensional space, they are

$$
\begin{aligned}
{}^2R - K^a{}_b K^b{}_a + K^2 &= {}^3R + 2\,{}^3R_{ab} n^a n^b \\
\Delta_a K_b{}^a - \Delta_b K &= -\,{}^3R_{cd} n^d h^c{}_b
\end{aligned}
$$

where ${}^2R = {}^2R_{AB} h^{AB}$ and $K = K_{AB} h^{AB}$. Written out these equations become respectively,

$$
\begin{aligned}
{}^{(2)}R - K^A{}_B K^B{}_A + K^2 &= \frac{2}{s}(\Delta_A \Delta^A s - sK\chi) + \frac{1}{2s^4}(\Delta_A w \Delta^A w + u^2) + 16\pi G\rho \\
\Delta_A K^A{}_B - \Delta_B K &= \Delta_B \chi + \frac{1}{s}\partial_B s\,\chi - \frac{1}{s} K_B{}^C \partial_C s + \frac{u}{2s^4}\partial_B w + 8\pi G j_B
\end{aligned}
$$

I have tried to follow Nakamura fairly closely in this and now seem to have agreement with his group's work.

We can write out now some of these quantities in terms of the metric functions

$$
\begin{aligned}
\Delta_A \Delta^A w &= \frac{1}{a^2 b}\left\{\partial_1\left[b(\partial_1 w - c\partial_2 w)\right] + \partial_2\left[-bc(\partial_1 w - c\partial_2 w) + \frac{1}{b}\,\partial_2 w\right]\right\} \\
\Delta_A s \Delta^A w &= \frac{1}{a^2}\left\{(\partial_1 s - c\partial_2 s)\,(\partial_1 w - c\partial_2 w) + \frac{1}{b^2}\partial_2 s\,\partial_2 w\right\}.
\end{aligned}
$$

Okay, so let's write out all the equations in all their scalar and to-be-differenced glory. Before any coordinate conditions or slicing conditions are imposed, we have 12 evolution equations and 3 constraint equations. The equations for the "pseudo-matter," (the scalar fields s and w) are

$$
\begin{aligned}
\partial_t s - \beta^1 \partial_1 s - \beta^2 \partial_2 s &= -\alpha s \chi \\
\partial_t \chi - \beta^1 \partial_1 \chi - \beta^2 \partial_2 \chi &= -\frac{\alpha}{sa^2 b}\left(\partial_1 p_s - c\partial_2 p_s - p_s \partial_2 c + \partial_2 q_s\right) + \alpha\chi(K + \chi) \\
&\quad + \frac{\alpha}{2s^4}\left(u^2 - \frac{1}{a^2}({p_w}^2 + {q_w}^2)\right) - \frac{1}{sa^2}(p_\alpha p_s + q_\alpha q_s) \\
\partial_t w - \beta^1 \partial_1 w - \beta^2 \partial_2 w &= -\alpha u \\
\partial_t u - \beta^1 \partial_1 u - \beta^2 \partial_2 u &= -\frac{\alpha}{a^2 b}\left(\partial_1 p_w - c\partial_2 p_w - p_w \partial_2 c + \partial_2 q_w\right) + \alpha u(K - 3\chi) \\
&\quad + \frac{3\alpha}{s}\frac{1}{a^2}(p_s p_w + q_s q_w) - \frac{1}{sa^2}(p_\alpha p_w + q_\alpha q_w)
\end{aligned}
$$

where we have made the convenient definition of the auxiliary variables

$$
\begin{aligned}
p_f &= b(\partial_1 f - c\partial_2 f) \\
q_f &= \frac{1}{b}\partial_2 f
\end{aligned}
$$

The subscripted f merely denotes what scalar function is being operated on. The equations for the evolution of the 2-metric functions are

$$
\begin{aligned}
\frac{1}{a}(\partial_t a - \beta^1 \partial_1 a - \beta^2 \partial_2 a) &= -\frac{\alpha}{a^2}\left(K_{11} - 2cK_{12} + cK_{22}\right) \\
&\quad + \partial_1 \beta^1 - c\partial_2 \beta^1 - b^2 c(\beta^1 \partial_1 c + \beta^2 \partial_2 c) \\
\frac{1}{b}(\partial_t b - \beta^1 \partial_1 b - \beta^2 \partial_2 b) &= \frac{\alpha}{a^2}\left(K_{11} - 2cK_{12} + cK_{22} - \frac{1}{b^2}K_{22}\right) \\
&\quad + c\partial_2 \beta^1 + \partial_2 \beta^2 - \partial_1 \beta^1 + c\partial_2 \beta^1 + b^2 c(\beta^1 \partial_1 c + \beta^2 \partial_2 c) \\
\partial_t c - 2\beta^1 \partial_1 c - 2\beta^2 \partial_2 c &= \frac{2\alpha}{a^2 b^2}(-K_{12} + cK_{22}) \\
&\quad + c(\partial_1 \beta^1 - c\partial_2 \beta^1) + \partial_1 \beta^2 - c\partial_2 \beta^2 + \frac{1}{b^2}\partial_2 \beta^1
\end{aligned}
$$

and the equations for the evolution of the extrinsic curvature are

$$
\begin{aligned}
(\partial_t - \beta^1\partial_1 - \beta^2\partial_2)K_1{}^1 &= K_2{}^1\partial_1\beta^2 - K_1{}^2\partial_2\beta^1 + \alpha\left[K_1{}^1(K+\chi) + {}^2R_1{}^1 - \frac{1}{s}\Delta_1\Delta^1 s\right] \\
&- \Delta_1\Delta^1\alpha - \frac{\alpha}{2s^4a^2}\left[p_w q_w - q_w{}^2 + u^2a^2\right] \\
&- 8\pi G\alpha\left[S_1{}^1 - \frac{1}{2}(S - \rho + \frac{1}{s^2}T_{\varphi\varphi})\right] \\
(\partial_t - \beta^1\partial_1 - \beta^2\partial_2)K_2{}^2 &= K_1{}^2\partial_2\beta^1 - K_2{}^1\partial_1\beta^2 + \alpha\left[K_2{}^2(K+\chi) + {}^2R_2{}^2 - \frac{1}{s}\Delta_2\Delta^2 s\right] \\
&- \Delta_2\Delta^2\alpha - \frac{\alpha}{2s^4a^2}\left[p_w{}^2 - cq_w p_w + u^2a^2\right] \\
&- 8\pi G\alpha\left[S_1{}^1 - \frac{1}{2}(S - \rho + \frac{1}{s^2}T_{\varphi\varphi})\right]
\end{aligned}
$$

Let us review what it is that we have done up to this point. We have started with GR coupled to a general matter field and assumed the presence of axisymmetry. We have divided out the axisymmetry and considered the resulting 3 dimensional theory. Then we have split time and space according to the ADM prescription. The resulting equations are 2 first order in time equations describing the scalar field s, 3 first order in time equations for the 3 dimensional "electromagnetic field," 3 first order in time equations for the evolution of the 2-metric, 3 first order in time equations for the evolution of the 2 dimensional extrinsic curvature and 3 constraint equations (Hamiltonian and 2 momentum constraints). To this point we have not made any coordinate or slicing conditions. In fact, we have yet to even choose a complete coordinate system. With regard to the latter point, we have chosen a time coordinate t for the 2+1 split and a coordinate φ (ignorable) adapted to our assumed axisymmetry. This seems to me about as general as we can be at this point. But in the following section, we will see that we have to start making some choices, particularly with regard to the choice of coordinate system – the regularity conditions would seem to demand it.

Regularity Conditions

I think I finally have something of a handle on the regularity conditions. The basic idea follows the review article by Bardeen and Piran and looks to adapt the Nakamura method to it. Bardeen and Piran argue that regularity can be enforced by demanding that locally, near the axis ($r \approx 0$), the *Cartesian components* of any tensor or vector can be expanded in non-negative powers of the Cartesian coordinates x, y, z. After finding the behavior near the axis, we transform back to our adapted coordinates and the near axis behavior can then be expressed in terms of our chosen coordinate system. Since we are enforcing what is in some sense a physical condition, we must work with the quantities on the full four dimensional manifold. The invariant way to define this is

$$\mathcal{L}_\xi \mathbf{Y} = 0$$

where ξ is the Killing vector in Cartesian coordinates near the axis and $\mathbf{Y}$ is any tensor quantity. We can write the Killing vector as

$$\xi = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

Taking the Lie derivative of a scalar quantity $\phi(t, x, y, z)$

$$-y\frac{\partial\phi}{\partial x} + x\frac{\partial\phi}{\partial y} = 0$$

shows us that $\phi = \phi(t, x^2 + y^2, z)$. This is of course no surprise as we would expect no dependance on φ.

As another example of this method, consider a vector η^α with the Lie derivative acting on it along the Killing vector. The equation for this is

$$\xi^\alpha \eta^\beta{}_{,\alpha} - \eta^\alpha \xi^\beta{}_{,\alpha} = 0$$

which written out in components yields four equations

$$-y\eta^t{}_{,x} + x\eta^t{}_{,y} = 0$$
$$-y\eta^x{}_{,x} + x\eta^x{}_{,y} = -\eta^y$$
$$-y\eta^y{}_{,x} + x\eta^y{}_{,y} = \eta^x$$
$$-y\eta^z{}_{,x} + x\eta^z{}_{,y} = 0.$$

The solution to these equations yields

$$\eta^t = g_1$$
$$\eta^x = xg_2 - yg_3$$
$$\eta^y = xg_3 + yg_2$$
$$\eta^z = g_4$$

where the g_i are functions of $t, x^2 + y^2$, and z. Transforming back to cylindrical coordinates (for example) using

$$\eta'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}\eta^\nu$$

we find

$$\eta^t = g_1(t, r^2, z)$$
$$\eta^r = rg_2(t, r^2, z)$$
$$\eta^z = g_4(t, r^2, z)$$
$$\eta^\varphi = g_3(t, r^2, z)$$

where the g_i's are expandable in non-negative powers of the arguments.

We can do the same thing for a general tensor $Y_{\mu\nu}$. For simplicity, we will assume that $Y_{\mu\nu}$ is symmetric. The equation for the Lie derivative on this tensor along the Killing vector is given in Bardeen and Piran as well as the resulting spatial equations written out in the Cartesian coordinate system. Let's go ahead and reproduce these equations together with those for the time components. The covariant equation is

$$Y_{\mu\nu,\lambda}\xi^\lambda + \xi^\lambda{}_{,\mu}Y_{\lambda\nu} + \xi^\lambda{}_{,\nu}Y_{\lambda\mu} = 0$$

where in terms of components we get

$$\begin{aligned} yY_{xx,x} - xY_{xx,y} &= 2Y_{xy} \\ yY_{yy,x} - xY_{yy,y} &= -2Y_{xy} \\ yY_{xy,x} - xY_{xy,y} &= Y_{yy} - Y_{xx} \end{aligned} \qquad \begin{aligned} yY_{zz,x} - xY_{zz,y} &= 0 \\ yY_{zx,x} - xY_{zx,y} &= Y_{zy} \\ yY_{zy,x} - xY_{zy,y} &= -Y_{zx} \end{aligned} \qquad \begin{aligned} yY_{tt,x} - xY_{tt,y} &= 0 \\ yY_{tx,x} - xY_{tx,y} &= Y_{ty} \\ yY_{ty,x} - xY_{ty,y} &= -Y_{tx} \\ yY_{tz,x} - xY_{tz,y} &= 0 \end{aligned}$$

The solution can now be found in terms of ten independent, regular functions f_i (with $i = 1\ldots 10$) which depend only on $t, x^2 + y^2$ and z.

$$\begin{aligned} Y_{xx} &= f_1 - 2xyf_2 + y^2 f_3 \\ Y_{yy} &= f_1 + 2xyf_2 + x^2 f_3 \\ Y_{xy} &= (x^2 - y^2)f_2 - xyf_3 \end{aligned} \qquad \begin{aligned} Y_{zz} &= f_4 \\ Y_{zx} &= xf_5 - yf_6 \\ Y_{zy} &= yf_5 + xf_6 \end{aligned} \qquad \begin{aligned} Y_{tt} &= f_7 \\ Y_{tx} &= xf_8 - yf_9 \\ Y_{ty} &= yf_8 + xf_9 \\ Y_{tz} &= f_{10} \end{aligned}$$

We now want to transform to the coordinates we would like to use. However, we can still keep things a bit general. We want to make a coordinate transformation from (t, x, y, z) to (t, x^1, x^2, ϕ) with x^1 and

x^2 being "arbitrary" curvilinear coordinates near the axis. The general coordinate transformation can be written in a suggestive notation as

$$
\begin{aligned}
x &= \bar{r}(x^1, x^2)\cos\varphi \\
y &= \bar{r}(x^1, x^2)\sin\varphi \\
z &= \bar{z}(x^1, x^2)
\end{aligned}
$$

with the inverse transformation, of course, assumed to exist. The problem with any coordinate system involving the cyclic coordinate φ is now fairly clear. At any points where the function $\bar{r}(x^1, x^2)$ goes to zero, the transformation becomes multi-valued since φ can take on any value between 0 and 2π. This occurs, of course, on the axis of symmetry and is the reason for our interest in the regularity conditions.

Using the usual transformation law,

$$
{Y'}_{\mu\nu} = \frac{\partial x^\lambda}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} Y_{\lambda\sigma}
$$

we find after expanding that

$$
\begin{aligned}
Y_{tt} &= f_7 \\
Y_{t\varphi} &= \bar{r}^2 f_9 \\
Y_{\varphi\varphi} &= \bar{r}^2 f_1 + \bar{r}^4 f_3 \\
Y_{tx^A} &= \bar{r}\frac{\partial \bar{r}}{\partial x^A} f_8 + \frac{\partial \bar{z}}{\partial x^A} f_{10} \\
Y_{\varphi x^A} &= \bar{r}^2 \frac{\partial \bar{z}}{\partial x^A} f_6 + \bar{r}^3 \frac{\partial \bar{r}}{\partial x^A} f_2 \\
Y_{x^A x^B} &= \frac{\partial \bar{r}}{\partial x^A}\frac{\partial \bar{r}}{\partial x^B} f_1 + \frac{\partial \bar{z}}{\partial x^A}\frac{\partial \bar{z}}{\partial x^B} f_4 \\
&\quad + \bar{r} f_5 \left[\frac{\partial \bar{r}}{\partial x^A}\frac{\partial \bar{z}}{\partial x^B} + \frac{\partial \bar{r}}{\partial x^B}\frac{\partial \bar{z}}{\partial x^A}\right].
\end{aligned}
$$

At this point, we have to make a choice in the coordinate system which we will use. We will choose to use cylindrical coordinates from here on out: $x^1 = r = \bar{r}$ and $x^2 = z = \bar{z}$. Another curvilinear coordinate system that might be interesting to consider would be spherical polar coordinates: $(x^1, x^2) = (r, \theta)$ and $\bar{r} = r\sin\theta$, $\bar{z} = r\cos\theta$. This is what Bardeen and Piran use, for example. Their transformations follow from our above equations. Given this general form of the transformation, though, we could also consider oblate or prolate spheroidal coordinates if we were ever crazy enough to do such a thing.

So, transforming the Cartesian spatial components to cylindrical coordinates and equating this tensor with $\gamma_{\mu\nu}$ allows us to determine the small r behavior of the full 4-metric. The transformation allows us to equate

$$
\begin{array}{ccccccccccc}
\gamma_{rr} & = & g_{rr} + s^2 A_r A_r & = & f_1 & , & \gamma_{\varphi\varphi} & = & s^2 & = & r^2 f_1 + r^4 f_3 \\
\gamma_{rz} & = & g_{rz} + s^2 A_r A_z & = & r f_5 & , & \gamma_{tt} & = & g_{tt} + s^2 A_t A_t & = & f_7 \\
\gamma_{r\varphi} & = & s^2 A_r & = & r^3 f_2 & , & \gamma_{tr} & = & g_{tr} + s^2 A_t A_r & = & r f_8 \\
\gamma_{zz} & = & g_{zz} + s^2 A_z A_z & = & f_4 & , & \gamma_{tz} & = & g_{tz} + s^2 A_t A_z & = & f_{10} \\
\gamma_{z\varphi} & = & s^2 A_z & = & r^2 f_6 & , & \gamma_{t\varphi} & = & s^2 A_t & = & r^2 f_9
\end{array}
$$

We can now untangle all of these and determine the behavior of the relevant quantities for small r

$$
\begin{array}{cc}
s^2 \approx r^2 & g_{rr} \approx C_2 \\
A_z \approx C_0 & A_t \approx C_3 \\
g_{zz} \approx C_1 & g_{tz} \approx C_4 \\
A_r \approx r & g_{tr} \approx r \\
g_{rz} \approx r & g_{tt} \approx C_5
\end{array}
$$

where the C_i's are "constant" with respect to r but will in general have t and z dependance. Using these relations, we can find the small r behavior for our particular parameterization of g_{ab}

$$\begin{aligned} a^2 &\approx C_6 \\ b^2 &\approx C_7 \\ c &\approx r \\ \beta^r &\approx r \\ \beta^z &\approx C_8 \end{aligned}$$

where again the C_i's are arbitrary functions of t and z.

We must also consider our potential function w. It is effectively a replacment for our A_a. Using the definition

$$\partial_a w = \frac{1}{2} s^3 \epsilon_{abc} g^{bd} g^{ce} (\partial_d A_e - \partial_e A_d)$$

we find that near $r = 0$

$$w \approx c_0 + r^4 F_1(t, r, z)$$

with c_0 a constant real number independent of t and z and F_1 a regular function of its arguments. Without any loss of generality, we can set $c_0 = 0$ since only the derivatives of w appear anywhere in our equations.

We must also consider the small r behavior of the components of the extrinsic curvature. To do this, we first need the relation between the "usual" extrinsic curvature ${}^3K_{\mu\nu}$ which comes from the ADM 3+1 prescription and the extrinsic curvature we have defined here ${}^2K_{ab}$ in the (2+1)+1 formulation. One way to examine this is to reverse the order of our decomposition. We first divided out the Killing vector X^μ and then split space and time. We could instead do the usual thing and perform the ADM split first and then consider the presence of the axisymmetry. In terms of the metric we would have

$$\begin{aligned} \gamma_{\mu\nu} &= g_{\mu\nu} + \frac{1}{s^2} X_\mu X_\nu \\ &= h_{\mu\nu} - n_\mu n_\nu + \frac{1}{s^2} X_\mu X_\nu \\ &= \mathcal{H}_{\mu\nu} - n_\mu n_\nu \end{aligned}$$

where we have defined the unit normal n_μ and a new purely spatial 3-metric $\mathcal{H}_{\mu\nu}$. The relevant relation can then be found as decomposing ${}^3K_{\mu\nu}$in terms of ${}^2K_{ab}$

$$\begin{aligned} {}^3K_{\mu\nu} &= -\nabla_\mu n_\nu \\ &= -\mathcal{H}_\mu{}^\alpha \mathcal{H}_\nu{}^\beta \nabla_\alpha n_\beta \\ &= {}^2K_{\mu\nu} - s^2 A_{(\mu} F_{\nu)\alpha} n^\alpha - s^2 A_\mu A_\nu \chi. \end{aligned}$$

This can be found in Nakamura's review article. The easiest way to verify the last line is to work backwards from it using the definitions we have made earlier.

Using our prototype tensor $Y_{\mu\nu}$, the behavior of the spatial components can now be associated with ${}^3K_{\mu\nu}$ from which we can deduce the he small r behavior for the components of ${}^2K_{ab}$ which we evolve

$$\begin{array}{ccccc} {}^3K_{rr} & = & d_1 & = & {}^2K_{rr} - s^2 A_r F_{r\alpha} n^\alpha + s^2 A_r A_r \chi \\ {}^3K_{r\varphi} & = & r^3 d_2 & = & -s^2 F_{r\alpha} n^\alpha + s^2 A_r \chi \\ {}^3K_{rz} & = & r d_5 & = & {}^2K_{rz} - s^2 A_{(r} F_{z)\alpha} n^\alpha + s^2 A_r A_z \chi \\ {}^3K_{zz} & = & d_4 & = & {}^2K_{zz} - s^2 A_z F_{z\alpha} n^\alpha + s^2 A_z A_z \chi \\ {}^3K_{z\varphi} & = & r^2 d_6 & = & s^2 F_{z\alpha} n^\alpha + s^2 A_z \chi \\ {}^3K_{\varphi\varphi} & = & r^2 d_1 + r^4 d_3 & = & s^2 \chi \end{array}$$

where the d_i with $i = 1 \ldots 6$ are independent, regular functions of t, r and z. Using these we find the small r behavior to be

$$\begin{aligned} \chi &\approx C_0 \\ {}^2K_{rr} &\approx C_1 \\ {}^2K_{rz} &\approx r \\ {}^2K_{zz} &\approx C_2 \end{aligned}$$

where the C_i's are different from before, but are again independent of r but have t and z dependance. In addition, we note that the "extra" conditions agree with the earlier behavior we found. Raising indices, we find finally

$$\begin{aligned} {}^2K_r{}^r &\approx \tilde{C}_1 \\ {}^2K_r{}^z &\approx r \\ {}^2K_z{}^z &\approx \tilde{C}_2. \end{aligned}$$

The Initial Value Problem

Okay, we have to face up to the complexity of the IVP (initial value problem) in GR. Fortunately, we have some things in our favor. Perhaps the most important is the observation that in our chosen coordinates (ρ, z) the 2-metric h_{AB} is conformally flat. This becomes crucial in simplifying the usual York-Lichnerowicz decomposition.

To begin our review of this decomposition, let us state that in this section, we will do things fairly generally. We start by conformally transforming a general spatial metric, γ_{ij} of dimension n as follows (in contrast with what we did before, i, j, k will run from 1 to n in this section)

$$\gamma_{ij} = \psi^{2p}\hat{\gamma}_{ij}, \qquad \gamma^{ij} = \psi^{-2p}\hat{\gamma}^{ij}$$

where all hatted quantities are in the conformally transformed manifold. The constant p is an integer which we are free to specify. It would seem fairly obvious that in our particular problem, we should equate the conformal factor with a^2 and we will do this eventually, but for the moment let's keep things somewhat general (so this can serve as a reminder for myself of the whole York procedure).

The Christoffel symbols now become with the conformal transformation

$$\Gamma^k_{ij} = \hat{\Gamma}^k_{ij} + p\left[\delta^k_j\partial_i \log\psi + \delta^k_i\partial_j \log\psi - \hat{\gamma}^{kl}\hat{\gamma}_{ij}\partial_l \log\psi\right]$$

and the Ricci tensor can now be written using the usual formula

$$\begin{aligned} R_{ij} = \hat{R}_{ij} &+ p(2-n)\hat{\nabla}_i\hat{\nabla}_j \log\psi - p\hat{\gamma}_{ij}\hat{\nabla}_i\hat{\nabla}^i \log\psi \\ &+ p^2(n-2)\left[\hat{\nabla}_i \log\psi\hat{\nabla}_j \log\psi - \hat{\gamma}_{ij}(\hat{\nabla}\log\psi)^2\right] \end{aligned}$$

from which we can get the Ricci scalar

$$\psi^{2p}R = \hat{R} + 2p(1-n)\hat{\nabla}_i\hat{\nabla}^i \log\psi + p^2(n-2)(1-n)(\hat{\nabla}\log\psi)^2.$$

Where again, in this section, we are using ∇_i as the covariant derivative constructed from γ_{ij} and $\hat{\nabla}_i$ as the covariant derivative constructed from the conformal metric $\hat{\gamma}_{ij}$. In the usual formulation, we look at the trace free part of the extrinsic curvature i.e. we subtract off the trace and give it a new name:

$$A_{ij} = K_{ij} - \frac{1}{n}\gamma_{ij}K$$

The momentum constraints

$$\nabla_j(K^{ij} - \gamma^{ij}K) = S^i$$

where S^i is the matter can now be written in terms of the symmetric and trace-free quantity A_{ij}:

$$\nabla_j A^{ij} = \frac{n-1}{n}\gamma^{ij}\,\nabla_j K + S^i$$

The burning question is how does the extrinsic curvature transform under the conformal transformation and by extension, how then do the constraints transform? The whole idea is one of simplification. We perform the conformal transformation in order that we can somehow simplify the constraints so that they are easier to solve. To this end, allow the extrinsic curvature to transform as

$$K^{ij} = \psi^q \hat{K}^{ij}$$

where q is another constant integer which we can choose and where one should note the upper indices (for some reason, working with the upper indices seems fairly standard). Using this, the fact that the trace of K_{ij} transforms as

$$K = K_i{}^i = \psi^{2p+q}\hat{K}$$

and that A^{ij} had better transform the same as does K^{ij}, we can rewrite the momentum constraints as

$$\hat{\nabla}_j\left[\hat{A}^{ij}\psi^{q+p(n+2)}\right] = \frac{n-1}{n}\psi^{q+p(n+2)}\hat{\gamma}^{ij}\left[(2p+q)\hat{K}\,\hat{\nabla}_j\log\psi + \hat{\nabla}_j\hat{K}\right] + \psi^{p(n+2)}S^i.$$

The easiest way to get this is simply to write out the constraints in terms of the tansformed quantities and transformed Christoffel symbols. A simple choice for the transformation properties of K^{ij} is that the covariant derivative be an invariant. In that case, we choose $q = -p(n+2)$. This is what we will do. (Another possible choice [that Evans uses] is that $tr(K)$ be a conformal invariant. For that, we would have $q = -2p$.)

Now, to solve the momentum constraints we decompose $\hat{A}^{ij}$ into transverse, $\hat{A}^{ij}_T$, and longitudinal, $\hat{A}^{ij}_L$, parts where by definition

$$\hat{\nabla}\hat{A}^{ij}_T = 0.$$

Both parts are separately traceless and symmetric. We can further introduce a "vector potential" W^i for the longitudinal part

$$\hat{A}^{ij}_L = \hat{\nabla}^i W^j + \hat{\nabla}^j W^i - \frac{2}{n}\hat{\gamma}^{ij}\hat{\nabla}_k W^k$$

Note that now we can write the conformally transformed covariant derivative of the conformally transformed trace-free part of the extrinsic curvature as (how's that for a mouthful?)

$$\begin{aligned}\hat{\nabla}_i\hat{A}^{ij} &= \hat{\nabla}_i\hat{\nabla}^i W^j + \hat{\nabla}_i\hat{\nabla}^j W^i - \frac{2}{n}\hat{\gamma}^{ij}\hat{\nabla}_i\hat{\nabla}_k W^k \\ &= \hat{\nabla}_i\hat{\nabla}^i W^j + \hat{R}^j{}_k W^k + \frac{n-2}{n}\hat{\gamma}^{ij}\hat{\nabla}_i\hat{\nabla}_k W^k\end{aligned}$$

Putting all of this together gives us the momentum constraints as a vector elliptic equation for W^i

$$\hat{\nabla}_j\hat{\nabla}^j W^i + \hat{R}^i{}_j W^j + \frac{n-2}{n}\hat{\gamma}^{ij}\hat{\nabla}_j\hat{\nabla}_k W^k = \frac{n-1}{n}\hat{\gamma}^{ij}\left[-pn\hat{K}\,\hat{\nabla}_j\log\psi + \hat{\nabla}_j\hat{K}\right] + \psi^{p(n+2)}S^i.$$

Note the simplification that results when the maximal slicing condition, $K = 0$, is used. Further simplification occurs when the spatial metric is 2 dimensional.

The Hamiltonian constraint is somewhat simpler fortunately. It is

$$R + K^2 - K_{ij}K^{ij} = 2\rho_H$$

Substituting in for our various quantities, we get

$$\begin{aligned}2p(n-1)\hat{\nabla}_i\hat{\nabla}^i\log\psi + p^2(n-1)(n-2)(\hat{\nabla}\log\psi)^2 = \hat{R} &+ \frac{n-1}{n}\psi^{2(q+3p)}\hat{K}^2 \\ &- \hat{A}_{ij}\hat{A}^{ij}\psi^{-2p(n-1)} - 2\psi^{2p}\rho_H.\end{aligned}$$

This is the general York scheme generalized to n dimensions. As it must, it reproduces the usual results for $n = 3$.

Now, however, let's consider $n = 2$ and our 2-metric is now h_{AB} and is conformally flat by our choice of coordinates. We had originally thought that we would choose $K = 0$. This corresponds to maximal slicing in the 2-manifold but something akin to what we'll call "axial" slicing in the 3-manifold since the usual extrinsic curvature now satisfies

$$\begin{aligned} {}^{(3)}K &= {}^{(2)}K + {}^{(3)}K_\varphi{}^\varphi \\ &= {}^{(3)}K_\varphi{}^\varphi \end{aligned}$$

This is in analogy to polar slicing where ${}^{(3)}K = K_r{}^r$. However, some testing of this slicing condition in spherical symmetry show that the resulting elliptic equation and boundary conditions may well be an ill-posed problem. The easiest way to see that is that the resulting differential operator in axial slicing becomes essentially a 2-d Laplacian rather than a 3-d Laplacian as in maximal slicing. This has a logarithm in its Green function which we should consider as its fundamental solution. What we want, though, is a solution to the lapse equation which should be asymptotically flat. Trying to impose such a boundary condition results in an ill-posed problem. So, we must choose a different slicing condition, and the most natural (or traditional) is thus maximal slicing.

With our conformally transformed 2-metric (which is flat) and our choice of cartesian coordinates (ρ, z), we have the following simplifications, namely that $\hat{R} = 0$, $\hat{\Gamma}^k_{ij} = 0$, $\hat{\nabla}_i = \partial_i$. Our above equations then reduce to

$$\begin{aligned} \hat{\nabla}_i\hat{\nabla}^i \log\psi^{2p} &= -\psi^{-2p}\hat{K}_{ij}\hat{K}^{ij} - 2\psi^{2p}\rho_H + \frac{1}{2}\psi^{-2p}\hat{K}^2 - \hat{A}_{ij}\hat{A}^{ij}\psi^{-2p} \\ \hat{\nabla}_i\hat{\nabla}^i W^j &= \psi^{4p}S^j + \frac{1}{2}\hat{\gamma}^{ij}\left[-\hat{K}\,\hat{\nabla}_j\log\psi^{2p} + \hat{\nabla}_j\hat{K}\right] \\ \hat{K}^{ij} &= \hat{K}^{ij}_T + \hat{K}^{ij}_L \end{aligned}$$

After all of that, this will probably be maddening, but this is not quite what we will do. The problem with the above is that in the usual formulation (i.e. $n = 3$), $\hat{A}^{ij}_T = \hat{K}^{ij}_T$ is freely specifiable initial data (e.g. let it be a gaussian). In $n = 2$, it is not. We have to find it from the constraints. But that is okay. We will not decompose K^{ij} as we have done. Instead, let us consider the evolution equation for the metric (again in n dimensions)

$$\partial_t\gamma_{ij} = -2\alpha K_{ij} + \nabla_i\beta_j + \nabla_j\beta_i$$

from which we can get

$$\partial_t(\log\gamma^{1/2}) = -\alpha K + \nabla_i\beta^i$$

where (this is a bit confusing) the quantity γ (without indices) is the determinant of γ_{ij} and K (without indices) is the trace of K_{ij}. This can be derived by contracting on the first equation and using the identity

$$\frac{1}{\gamma}\partial_t\gamma = \gamma^{ij}\partial_t\gamma_{ij}.$$

Now, making a non-obvious combination of these (actually it's the trace-free thing again), we can write

$$\partial_t\gamma_{ij} = -2\alpha(K_{ij} - \frac{1}{n}\gamma_{ij}K) + \nabla_i\beta_j + \nabla_j\beta_i + \frac{2}{n}\gamma_{ij}(-\alpha K + \nabla_k\beta^k) - \frac{2}{n}\gamma_{ij}\nabla_k\beta^k$$

which, on rearrangement yields

$$\gamma^{1/n}\partial_t\left[\gamma^{-1/n}\gamma_{ij}\right] = -2\alpha(K_{ij} - \frac{1}{n}\gamma_{ij}K) + \nabla_i\beta_j + \nabla_j\beta_i - \frac{2}{n}\gamma_{ij}\nabla_k\beta^k$$

The point of all this is that the quantity in brackets is conformally invariant:

$$\gamma^{-1/n}\gamma_{ij} = \hat{\gamma}^{-1/n}\hat{\gamma}_{ij}$$

and will be a constant because of our choice of coordinates and since our conformal metric is flat. Thus the time derivative and hence the left hand side above is zero. Note that this is exactly what Wilson and Matthews do in their "conformal flatness approximation" scheme. There they assume that the conformal metric is always flat (which they argue is good provided gravitational radiation is negligible) and solve a simplified set of equations for the inspiral of two neutron stars. Here however (in 2 spatial dimensions), the flatness of the conformal metric is *exact.*

Let us now revert to our previous notation and work with our two dimensional metric h_{AB}. The basic idea is to use the covariant derivative operators associated with the conformal metric $\hat{h}_{AB}$ but *not* to use other conformally transformed quantities. This may seem a bit mixed up, but it is effectively what is done in the usual initial value formulation. One transforms to the conformal metric to simplify the equations and then at the end of the day transforms back to the "physical" quantities associated with the physical metric. If we can take advantage of the simplifications inherent in the conformal metric without transforming everything to their conformal equivalents, why not?

In addition, it seems to me that by doing this, we get the added bonus (actually it is more by virtue again of our 2 dimensional spatial metric) that there is no gravitational radiation in the initial data but what we explicitly give. Note that earlier I made mention of the fact that in the York scheme, $\hat{A}_T^{ij} = \hat{K}_T^{ij}$ is freely specifiable initial data. One choice of initial data of course would be no initial gravitational radiation. However, how does one specify that? A natural choice would seem to be flat initial data with $\hat{\gamma}_{ij} = 0$ and $\hat{K}_T^{ij} = 0$, but it turns out that the conformal transformation back to the physical metric mixes the components of the decomposition of K^{ij} such that the longitudinal part of it will pick up some radiative parts and will still contain preexisting gravitational radiation. This does not seem to be true in our scheme and it seems that if we want preexisting radiation or not we can say so simply by our conditions on the radiative variables s, χ, u and w.

Anyway, continuing on we will use maximal slicing in the 3-manifold, ${}^3K = 0$. The extrinsic curvature is now be given by

$$\begin{aligned}
K^{AB} &= \frac{1}{2}h^{AB}K + \frac{1}{2\alpha}\left[\Delta^A\beta^B + \Delta^B\beta^A - h^{AB}\Delta_C\beta^C\right] \\
&= \frac{1}{2}h^{AB}K + \frac{\psi^{-2p}}{2\alpha}\left[\hat{\Delta}^A\beta^B + \hat{\Delta}^B\beta^A - \hat{h}^{AB}\hat{\Delta}_C\beta^C\right]
\end{aligned}$$

The momentum constraints, $\Delta_A K^{AB} - \Delta^B K = S^B$, can now be written in terms of the shift vector

$$\begin{aligned}
\hat{\Delta}_A\hat{\Delta}^A\beta^B &= 2\alpha\psi^{2p}S^B + \partial_A(\log\alpha\psi^{-2p})\left\{\hat{\Delta}^A\beta^B + \hat{\Delta}^B\beta^A - \hat{h}^{AB}\hat{\Delta}_C\beta^C\right\} \\
&+ \frac{1}{2}(3\hat{\Delta}^B K + K\hat{\Delta}^B\log\psi^{2p})
\end{aligned}$$

where the "matter" part is

$$S_B = \partial_B\chi + \frac{1}{s}\chi\partial_B s - \frac{1}{s}K_B{}^C\partial_C s + \frac{u}{2s^4}\partial_B w + 8\pi G j_B$$

Again, we note that the covariant derivative operator $\hat{\Delta}_A$ is built out of the conformal metric $\hat{h}_{AB}$. In general, this metric will be our earlier version of h_{AB} with the a^2 factors divided out. In the particular case of cylindrical or spherical polar coordinates, the conformal metric will be

$$\hat{h}_{AB} = \begin{pmatrix} 1 & 0 \\ 0 & b(x^1)^2 \end{pmatrix}$$

with $b(x^1) = 1$ for cylindrical coordinates and $b(x^1) = b(r) = r$ for spherical polar coordinates.

The Hamiltonian constraint is now

$$\hat{\Delta}_A\hat{\Delta}^A\log\psi^{2p} = -2\rho_H\psi^{2p} - \psi^{2p}K_{AB}K^{AB}$$

where

$$K^{AB} = \frac{1}{2}h^{AB}K + \frac{\psi^{-2p}}{2\alpha}\left(\hat{\Delta}^A\beta^B + \hat{\Delta}^B\beta^A - \hat{h}^{AB}\hat{\Delta}_C\beta^C\right)$$

and the slicing condition becomes

$$\hat{\Delta}_A\hat{\Delta}^A\alpha = -(\log s)_{,A}\alpha_,{}^A + \alpha\psi^{2p}\left\{(K+\chi)^2 - n^a(K+\chi)_{,a}\right\}$$
$$-\alpha\left[\frac{2}{s}\hat{\Delta}_A\hat{\Delta}^A s - \hat{\Delta}_A\hat{\Delta}^A\log\psi^{2p} - \frac{\psi^{2p}}{2s^4}u^2 - 8\pi G\,\psi^{2p}(\rho - \frac{1}{s^2}T_{\varphi\varphi})\right]$$

The Equations to be Differenced (Finally)

The following are now the axisymmetric Einstein equations coupled to a general matter field that obeys the conditions we stated earlier. There are four evolution equations for the two radiative parts of the metric, two (complex) evolution equations for the complex scalar field and four constraint equations determining the kinematic variables. The slicing equation for the lapse is written suggestively for the imposition of maximal slicing, though in principle, a different slicing could be used.

$$\dot{s} - \beta^A\partial_A s = -\alpha s\chi$$
$$\dot{\chi} - \beta^A\partial_A\chi = -\frac{1}{sa^2}\hat{\Delta}_A(\alpha\hat{\Delta}^A s) + \alpha\chi(K+\chi)$$
$$+\frac{\alpha}{2a^2s^4}(a^2u^2 - \hat{\Delta}_A w\hat{\Delta}^A w) - 8\pi G\frac{\alpha}{s^2}\left(T_{\varphi\varphi} - \frac{1}{2}s^2\,{}^{(4)}T\right)$$
$$\dot{w} - \beta^A\partial_A w = -\alpha u$$
$$\dot{u} - \beta^A\partial_A u = -\frac{s^3}{a^2}\hat{\Delta}_A\left(\frac{\alpha}{s^3}\hat{\Delta}^A w\right) + \alpha u(K - 3\chi)$$

$$\hat{\Delta}_A\hat{\Delta}^A(\log a^2) = -\frac{2}{s}\hat{\Delta}_A\hat{\Delta}^A s - \frac{a^2}{2\alpha^2}\left[\hat{\Delta}_A\beta^B\,\hat{\Delta}_B\beta^A + \hat{h}_{BC}\hat{\Delta}_A\beta^B\,\hat{\Delta}^A\beta^C - (\hat{\Delta}_A\beta^A)^2\right]$$
$$+\frac{a^2}{2}K(K+4\chi) - \frac{1}{2s^4}(\hat{\Delta}_A w\hat{\Delta}^A w + a^2u^2) - 16\pi G a^2 T_{\mu\nu}n^\mu n^\nu$$
$$\hat{\Delta}_A\hat{\Delta}^A\beta^B = \alpha s\hat{\Delta}^B\left[\frac{1}{s}(K+\chi)\right] + \hat{\Delta}_A\left(\log\frac{\alpha}{sa^2}\right)\left\{\hat{\Delta}^A\beta^B + \hat{\Delta}^B\beta^A - \hat{h}^{AB}\hat{\Delta}_C\beta^C\right\}$$
$$+\alpha\hat{\Delta}^B\chi + 3\alpha\chi\hat{\Delta}^B(\log s) + \frac{\alpha u}{s^4}\hat{\Delta}^B w - 16\pi G\alpha T_{bA}n^b\hat{h}^{AB}$$
$$\hat{\Delta}_A\hat{\Delta}^A\alpha = \alpha a^2\left\{(K+\chi)^2 - n^a(K+\chi)_{,a}\right\} - \alpha\hat{\Delta}_A\hat{\Delta}^A\log a^2$$
$$-\frac{2\alpha}{s}\hat{\Delta}_A\hat{\Delta}^A s - \frac{1}{s}\hat{\Delta}_A s\hat{\Delta}^A\alpha - \frac{\alpha a^2}{2s^4}u^2 - 8\pi G\alpha a^2\left(T_{\mu\nu}n^\mu n^\nu - \frac{1}{2}\,{}^{(4)}T\right)$$

where these have to be supplemented by equations of motion for the matter. Note that we could have also written $\rho_H = T_{\mu\nu}n^\mu n^\nu$ and $j^B = -T_{bA}n^b\hat{h}^{AB}$ using our earlier definitions.

Let's now consider the particular example of a nonlinear sigma model. The action for the complete theory is then given by

$$S = \int d^4x\sqrt{-g}\left\{\frac{R}{16\pi G} - \frac{|\nabla f|^2}{(1-\kappa|f|^2)^2} - V(f,f^*)\right\},$$

The corresponding equations are now

$$\dot{s} - \beta^A\partial_A s = -\alpha s\chi$$
$$\dot{\chi} - \beta^A\partial_A\chi = -\frac{1}{sa^2}\hat{\Delta}_A(\alpha\hat{\Delta}^A s) + \alpha\chi(K+\chi)$$
$$+\frac{\alpha}{2a^2s^4}(a^2u^2 - \hat{\Delta}_A w\hat{\Delta}^A w) - 8\pi G\alpha V$$
$$\dot{w} - \beta^A\partial_A w = -\alpha u$$
$$\dot{u} - \beta^A\partial_A u = -\frac{s^3}{a^2}\hat{\Delta}_A\left(\frac{\alpha}{s^3}\hat{\Delta}^A w\right) + \alpha u(K - 3\chi)$$

$$
\begin{aligned}
\dot{f} - \beta^A \partial_A f &= -\alpha F \\
\dot{F} - \beta^A \partial_A F &= -\frac{1}{sa^2}\hat{\Delta}_A\left(\alpha s \hat{\Delta}^A f\right) + \alpha F(K+\chi) \\
&\quad + \frac{2\kappa\alpha f^*}{a^2(1-\kappa|f|^2)}(a^2F^2 - \hat{\Delta}_A f \hat{\Delta}^A f) + \alpha(1-\kappa|f|^2)^2\frac{\partial V}{\partial f^*}
\end{aligned}
$$

$$
\begin{aligned}
\hat{\Delta}_A\hat{\Delta}^A(\log a^2) &= -\frac{2}{s}\hat{\Delta}_A\hat{\Delta}^A s - \frac{a^2}{2\alpha^2}\left[\hat{\Delta}_A\beta^B\,\hat{\Delta}_B\beta^A + \hat{h}_{BC}\hat{\Delta}_A\beta^B\,\hat{\Delta}^A\beta^C - (\hat{\Delta}_A\beta^A)^2\right] \\
&\quad + \frac{a^2}{2}K(K+4\chi) - \frac{1}{2s^4}(\hat{\Delta}_A w\hat{\Delta}^A w + u^2a^2) \\
&\quad - 16\pi G\left\{\frac{a^2|F|^2 + \hat{\Delta}_A f\hat{\Delta}^A f^*}{(1-\kappa|f|^2)^2} + a^2V\right\}
\end{aligned}
$$

$$
\begin{aligned}
\hat{\Delta}_A\hat{\Delta}^A\beta^B &= \alpha s\hat{\Delta}^B\left[\frac{1}{s}(K+\chi)\right] + \hat{\Delta}_A\left(\log\frac{\alpha}{sa^2}\right)\left\{\hat{\Delta}^A\beta^B + \hat{\Delta}^B\beta^A - \hat{h}^{AB}\hat{\Delta}_C\beta^C\right\} \\
&\quad + \alpha\hat{\Delta}^B\chi + 3\alpha\chi\hat{\Delta}^B(\log s) + \frac{\alpha u}{s^4}\hat{\Delta}^B w + 16\pi G\alpha\left\{\frac{F\hat{\Delta}^B f^* + F^*\hat{\Delta}^B f}{(1-\kappa|f|^2)^2}\right\}
\end{aligned}
$$

$$
\begin{aligned}
\hat{\Delta}_A\hat{\Delta}^A\alpha &= \alpha a^2\left\{(K+\chi)^2 - n^a(K+\chi)_{,a}\right\} - \alpha\hat{\Delta}_A\hat{\Delta}^A\log a^2 \\
&\quad - \frac{2\alpha}{s}\hat{\Delta}_A\hat{\Delta}^A s - \frac{1}{s}\hat{\Delta}_A s\hat{\Delta}^A\alpha - \frac{\alpha a^2}{2s^4}u^2 - 16\pi G\alpha\left\{\frac{\hat{\Delta}_A f\hat{\Delta}^A f^*}{(1-\kappa|f|^2)^2} + \frac{3}{2}a^2V\right\}
\end{aligned}
$$

where the derivative operator $\hat{\Delta}_A$ is built out of $\hat{h}_{AB} = \mathrm{diag}\left(1, b(x^1)\right)$, K is the extrinsic curvature of the 2-manifold, $|\nabla f|^2 = \nabla_a f\nabla^a f^*$, the indices A and B run over 1 and 2, and the indices a and b run over 0, 1, and 2.

We now take these equations and impose maximal slicing: ${}^3K = K + \chi = 0$ and use cylindrical coordinates $b = 1$. The resulting equations are

$$
\begin{aligned}
\dot{s} - \beta^\rho s_{,\rho} - \beta^z s_{,z} &= -\alpha s\chi \\
\dot{\chi} - \beta^\rho\chi_{,\rho} - \beta^z\chi_{,z} &= -\frac{1}{sa^2}\left[(\alpha s_{,\rho})_{,\rho} + (\alpha s_{,z})_{,z}\right] + \frac{\alpha}{2a^2s^4}(a^2u^2 - w_{,\rho}{}^2 - w_{,z}{}^2) - 8\pi G\alpha V \\
\dot{w} - \beta^\rho w_{,\rho} - \beta^z w_{,z} &= -\alpha u \\
\dot{u} - \beta^\rho u_{,\rho} - \beta^z u_{,z} &= -\frac{s^3}{a^2}\left[\left(\frac{\alpha}{s^3}w_{,\rho}\right)_{,\rho} + \left(\frac{\alpha}{s^3}w_{,z}\right)_{,z}\right] - 4\alpha u\chi
\end{aligned}
$$

$$
\begin{aligned}
\dot{f} - \beta^\rho f_{,\rho} - \beta^z f_{,z} &= -\alpha F \\
\dot{F} - \beta^\rho F_{,\rho} - \beta^z F_{,z} &= -\frac{1}{sa^2}\left[(\alpha s f_{,\rho})_{,\rho} + (\alpha s f_{,z})_{,z}\right] \\
&\quad + \frac{2\kappa\alpha f^*}{a^2(1-\kappa|f|^2)}(a^2F^2 - f_{,\rho}{}^2 - f_{,z}{}^2) + \alpha(1-\kappa|f|^2)^2\frac{\partial V}{\partial f^*}
\end{aligned}
$$

$$
\begin{aligned}
(\log a^2)_{,\rho\rho} + (\log a^2)_{,zz} &= -\frac{2}{s}(s_{,\rho\rho} + s_{,zz}) - \frac{a^2}{2\alpha^2}\left[(\beta^\rho{}_{,\rho} - \beta^z{}_{,z})^2 + (\beta^z{}_{,\rho} + \beta^\rho{}_{,z})^2\right] - \frac{3}{2}a^2\chi^2 \\
&\quad - \frac{1}{2s^4}(w_{,\rho}{}^2 + w_{,z}{}^2 + u^2a^2) - 16\pi G\left\{\frac{a^2|F|^2 + |f_{,\rho}|^2 + |f_{,z}|^2}{(1-\kappa|f|^2)^2} + a^2V\right\}
\end{aligned}
$$

$$
\begin{aligned}
\beta^\rho{}_{,\rho\rho} + \beta^\rho{}_{,zz} &= \left(\log\frac{\alpha}{a^2s}\right)_{,\rho}(\beta^\rho{}_{,\rho} - \beta^z{}_{,z}) + \left(\log\frac{\alpha}{a^2s}\right)_{,z}(\beta^z{}_{,\rho} + \beta^\rho{}_{,z}) \\
&\quad + \alpha\chi_{,\rho} + 3\alpha\chi\frac{s_{,\rho}}{s} + \frac{\alpha u}{s^4}w_{,\rho} + 16\pi G\alpha\frac{Ff^*{}_{,\rho} + F^*f_{,\rho}}{(1-\kappa|f|^2)^2}
\end{aligned}
$$

$$\beta^z{}_{,\rho\rho} + \beta^z{}_{,zz} = \left(\log\frac{\alpha}{a^2 s}\right)_{,\rho}(\beta^z{}_{,\rho} + \beta^\rho{}_{,z}) + \left(\log\frac{\alpha}{a^2 s}\right)_{,z}(\beta^z{}_{,z} - \beta^\rho{}_{,\rho})$$
$$+ \alpha\chi_{,z} + 3\alpha\chi\frac{s_{,z}}{s} + \frac{\alpha u}{s^4}w_{,z} + 16\pi G\alpha\frac{Ff^*{}_{,z} + F^* f_{,z}}{(1-\kappa|f|^2)^2}$$

$$\alpha_{,\rho\rho} + \alpha_{,zz} = -\alpha\left[(\log a^2)_{,\rho\rho} + (\log a^2)_{,zz}\right] - \frac{2\alpha}{s}(s_{,\rho\rho} + s_{,zz})$$
$$- \frac{1}{s}(s_{,\rho}\alpha_{,\rho} + s_{,z}\alpha_{,z}) - \frac{\alpha a^2}{2s^4}u^2 - 16\pi G\alpha\left\{\frac{|f_{,\rho}|^2 + |f_{,z}|^2}{(1-\kappa|f|^2)^2} + \frac{3}{2}a^2 V\right\}$$

Knowing that as $\rho \to 0$, we have $s \sim \rho$, $\beta^\rho \sim \rho$, $u \sim \rho^4$, and $w \sim \rho^4$, we can now use regularized variables via the substitutions: $s = \rho\bar{s}$, $\beta^\rho = \rho\,\bar{\beta}^\rho$, $u = \rho^4\bar{u}$, and $w = \rho^4\bar{w}$. However at the risk of a great deal of confusion, I will **NOT** explicitly write out all the barred quantities. So all the s, β^ρ, u and w's in the following should be considered barred. Sorry for the confusion, but it's easier for me at this point.

$$\dot{s} - \rho\beta^\rho s_{,\rho} - \beta^z s_{,z} = -\alpha s\chi + \beta^\rho s$$
$$\dot{\chi} - \rho\beta^\rho\chi_{,\rho} - \beta^z\chi_{,z} = -\frac{1}{sa^2}\left[(\alpha s_{,\rho})_{,\rho} + (\alpha s_{,z})_{,z}\right] - \frac{1}{\rho s^2 a^2}(\alpha s^2)_{,\rho}$$
$$- \frac{\rho^2\alpha}{2a^2 s^4}\left[(4w + \rho w_{,\rho})^2 + (\rho w_{,z})^2 - \rho^2 a^2 u^2\right] - 8\pi G\alpha V$$
$$\dot{w} - \rho\beta^\rho w_{,\rho} - \beta^z w_{,z} = -\alpha u + 4\beta^\rho w$$
$$\dot{u} - \rho\beta^\rho u_{,\rho} - \beta^z u_{,z} = -\frac{s^3}{a^2}\left[\frac{1}{\rho}\left(\frac{\alpha}{s^3}\rho w_{,\rho}\right)_{,\rho} + \left(\frac{\alpha}{s^3}w_{,z}\right)_{,z}\right]$$
$$- \frac{4s^3}{\rho a^2}\left(\frac{\alpha w}{s^3}\right)_{,\rho} - 4\alpha u\chi + 4\beta^\rho u$$

$$\dot{f} - \rho\beta^\rho f_{,\rho} - \beta^z f_{,z} = -\alpha F$$
$$\dot{F} - \rho\beta^\rho F_{,\rho} - \beta^z F_{,z} = -\frac{1}{sa^2}\left[(\alpha s f_{,\rho})_{,\rho} + (\alpha s f_{,z})_{,z}\right] - \frac{\alpha}{\rho a^2}f_{,\rho}$$
$$+ \frac{2\kappa f^*}{a^2(1-\kappa|f|^2)^2}(a^2F^2 - f_{,\rho}{}^2 - f_{,z}{}^2) + \alpha(1-\kappa|f|^2)^2\frac{\partial V}{\partial f^*}$$

$$(\log a^2)_{,\rho\rho} + (\log a^2)_{,zz} = -\frac{2}{s}\left[s_{,\rho\rho} + \frac{2}{\rho}s_{,\rho} + s_{,zz}\right] - \frac{a^2}{2\alpha^2}\left[((\rho\beta^\rho)_{,\rho} - \beta^z{}_{,z})^2 + (\beta^z{}_{,\rho} + \rho\beta^\rho{}_{,z})^2\right]$$
$$- \frac{3}{2}a^2\chi^2 - \frac{\rho^2}{2s^4}\left[(4w + \rho w_{,\rho})^2 + (\rho w_{,z})^2 + (\rho u a)^2\right]$$
$$- 16\pi G\left\{\frac{a^2|F|^2 + |f_{,\rho}|^2 + |f_{,z}|^2}{(1-\kappa|f|^2)^2} + a^2 V\right\}$$

$$\beta^\rho{}_{,\rho\rho} + \frac{2}{\rho}\beta^\rho{}_{,\rho} + \beta^\rho{}_{,zz} = \frac{1}{\rho}\left(\log\frac{\alpha}{a^2 s}\right)_{,\rho}(\rho\beta^\rho{}_{,\rho} + \beta^\rho - \beta^z{}_{,z}) + \left(\log\frac{\alpha}{a^2 s}\right)_{,z}(\frac{1}{\rho}\beta^z{}_{,\rho} + \beta^\rho{}_{,z})$$
$$- \frac{1}{\rho^2}(\rho\beta^\rho{}_{,\rho} + \beta^\rho - \beta^z{}_{,z}) + \frac{1}{\rho}\alpha\chi_{,\rho} + 3\alpha\chi\left(\frac{1}{\rho^2} + \frac{s_{,\rho}}{\rho s}\right)$$
$$+ \frac{\alpha u}{s^4}\rho^2(4w + \rho w_{,\rho}) + 16\pi G\alpha\frac{1}{\rho}\frac{Ff^*{}_{,\rho} + F^* f_{,\rho}}{(1-\kappa|f|^2)^2}$$

$$\beta^z{}_{,\rho\rho} + \beta^z{}_{,zz} = \left[\left(\log\frac{\alpha}{a^2 s}\right)_{,\rho} - \frac{1}{\rho}\right](\beta^z{}_{,\rho} + \rho\beta^\rho{}_{,z}) + \left(\log\frac{\alpha}{a^2 s}\right)_{,z}(\beta^z{}_{,z} - (\rho\beta^\rho)_{,\rho})$$
$$+ \alpha\chi_{,z} + 3\alpha\chi\frac{s_{,z}}{s} + \frac{\alpha u}{s^4}\rho^4 w_{,z} + 16\pi G\alpha a^2\frac{Ff^*{}_{,z} + F^* f_{,z}}{(1-\kappa|f|^2)^2}$$

$$\alpha_{,\rho\rho} + \frac{1}{\rho}\alpha_{,\rho} + \alpha_{,zz} = -\frac{2\alpha}{s}\left[s_{,\rho\rho} + \frac{2}{\rho}s_{,\rho} + s_{,zz}\right] - \alpha\left[(\log a^2)_{,\rho\rho} + (\log a^2)_{,zz}\right]$$
$$- \frac{\alpha a^2}{2s^4}u^2\rho^4 - \frac{1}{s}(s_{,\rho}\alpha_{,\rho} + s_{,z}\alpha_{,z}) - 16\pi G\alpha\left\{\frac{|f_{,\rho}|^2 + |f_{,z}|^2}{(1-\kappa|f|^2)^2} + \frac{3}{2}a^2V\right\}$$

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